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Asymptotically Degenerate Maximum Eigenvalues of the Eight-Vertex Model Transfer Matrix and Interfacial Tension

Rodney Baxter[1]

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We obtain complex nonlinear integral equations for the two asymptotically degenerate maximum eigenvalues of the transfer matrix of the eight-vertex model. These are exact for a lattice of a finite number N of columns. Solving the equations recursively gives an expansion of the eigenvalues about $N = \infty$. Thus we can obtain the interfacial tension of the model, as well as rederiving our previous result for the free energy.

KEY WORDS: Eight-vertex model; finite size effects; interfacial tension; scaling laws; transfer matrix; eigenvalues.

1. INTRODUCTION

In previous papers[1,2] we have obtained expressions for the eigenvalues and eigenvectors of the transfer matrix **T** of the zero-field, two-dimensional, eight-vertex model in lattice statistics. Since **T** commutes with the Hamiltonian $\mathscr{H}$ of the one-dimensional XYZ chain (for appropriate values of the interaction parameters), we can also obtain the eigenvalues and eigenvectors of $\mathscr{H}$.

For a lattice of N columns (or a chain of N sites) the results are expressed in terms of $n = \frac{1}{2}N$ parameters $v_1,..., v_n$ or $\phi_1,..., \phi_n$, which must be obtained by solving n simultaneous transcendental equations. This is the same

[1] Research School of Physical Sciences, The Australian National University, Canberra, Australia.

situation as that which occurs in the Bethe ansatz solutions of the ice-type lattice models,[3] or the Heisenberg chain.[4] These are special cases of our models.

Until now explicit solutions of these equations have only been obtained in the limit of N and n infinitely large. More precisely, if $\Lambda_{\max}$ is the numerically largest eigenvalue of **T**, the limit

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln \Lambda_{\max} \tag{1}$$

has been evaluated,[1] giving f, which is the free energy per site of the infinite lattice (β here is the Boltzmann factor $1/kT$). However, the way in which this limit is approached has until now not been calculated.

In particular, for ordered states[2] of the system we expect the two numerically largest eigenvalues Λ_0 and Λ_1 of **T** to be asymptotically degenerate in the limit of N large, i.e.,

$$\Lambda_0/\Lambda_1 = \pm[1 + \mathcal{O}(e^{-N\beta\sigma})] \tag{2}$$

where σ is a positive quantity that can be identified with the interfacial tension.[5]

In this paper we derive complex nonlinear integral equations that determine Λ_0 or Λ_1 for *finite N*. This formulation is appropriate for examining the behavior as $N \to \infty$, since solving the equations by recursion gives to first order the free energy f as defined by (1), and then to second order the dominant, but exponentially small, corrections to Λ_0 or Λ_1. Thus we find that

$$\Lambda_i = e^{-N\beta f}(1 + \epsilon_i), \qquad i = 0 \quad \text{or} \quad 1 \tag{3}$$

where ϵ_i decays exponentially with increasing N and we can evaluate

$$\lim_{N\to\infty} N^{-1} \ln \epsilon_i \tag{4}$$

Proceeding in this way, we are able to verify (2) and calculate σ. We then compare the behavior of f and σ near the critical temperature and find that certain predictions of scaling theory are satisfied.

The integral equations themselves are of some interest in that they are of a similar form to those found by Gaudin,[6] Takahashi,[7] and Johnson and McCoy[8] for the partition functions of the Heisenberg and XYZ chains. (However, we have basically just one complex equation as against their

[2] From the weak-graph symmetry [Eq. (12) of Ref. 9) Λ_0 and Λ_1 must also be asymptotically degenerate even in the disordered state. However, one eigenvector lies in the subspace with an even number of down arrows per row of the lattice, the other in the subspace with an odd number of down arrows. Thus they do not interact with one another.

hierarchies of real equations.) That this should be so is perhaps not surprising, since a knowledge of the maximum eigenvalue of the row transfer matrix for all finite N presumably gives considerable information on the distribution of eigenvalues of the column transfer matrix for an infinite number of rows. The partition function of the XYZ chain is simply an average over this distribution.

We give our results in the following sections of this paper. In Section 2 we obtain some symmetry relations that enable us to consider just one of the four possible ordered states of the system. In Section 3 we quote our previous results[1] for the equations that determine the eigenvalues of $\mathbf{T}$, making some notational changes to suit our present purposes. In Section 4 we show how we can transform these to the complex integral equations mentioned above. In Section 5 we consider the limit $N \to \infty$ and obtain f and σ. In Section 6 we give the behavior of these functions near the critical temperature, and in Section 7 we show that this behavior agrees with scaling theory.

As far as possible detailed working is left to the appendices.

2. SYMMETRY RELATIONS

In the eight-vertex model there are eight possible configurations of arrows at each vertex, occurring in four pairs with Boltzmann weights a, b, c, d. The system assumes an ordered state if one of the weights is greater than the sum of the other three, e.g., if

$$c > a + b + d \tag{5}$$

There are therefore four possible ordered states, depending on which of the Boltzmann weights is greater. However, for fixed interaction energies only one of the Boltzmann weights (the one of lowest energy) can dominate the others, so at most one phase transition can occur as the temperature T decreases from $+\infty$ to zero.

Fortunately there are symmetry relations[9] that enable us to consider only one of the ordered states, without any loss of generality. From the definition of the transfer matrix $\mathbf{T}$ given in Eqs. (3.5) and (3.6) of Ref. 1, we see that the simultaneous interchanges

$$a \leftrightarrow b, \qquad c \leftrightarrow d \tag{6}$$

leave all elements of $\mathbf{T}$ unchanged. Thus each eigenvalue Λ of $\mathbf{T}$ satisfies the symmetry relation

$$\Lambda \equiv \Lambda(a, b \mid c, d) = \Lambda(b, a \mid d, c) \tag{7}$$

Further, let σ_J^x be the Pauli operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

acting on the arrow, or "spin," on column J of the lattice. Take the number N of columns to be even and define

$$\mathbf{P}_o = \sigma_1^x \sigma_3^x \sigma_5^x \cdots \sigma_{N-1}^x, \qquad \mathbf{P}_e = \sigma_2^x \sigma_4^x \sigma_6^x \cdots \sigma_N^x \tag{9}$$

Then from Eqs. (3.5) and (3.6) of Ref. 1 we find that

$$\mathbf{T} \equiv \mathbf{T}(a, b \mid c, d) = \mathbf{P}_o \mathbf{T}(c, d \mid a, b)\, \mathbf{P}_e \tag{10}$$

The operator $\mathbf{P}_o\mathbf{P}_e$ simply reverses all arrows in a row and commutes with $\mathbf{T}$. Consider an eigenvector $\mathbf{x}$ of $\mathbf{T}$ and let $\nu = 0$ or 1 according to whether $\mathbf{x}$ is symmetric or antisymmetric with respect to reversing all arrows. Then

$$\mathbf{P}_o\mathbf{P}_e\mathbf{x} = (-1)^\nu \mathbf{x} \tag{11}$$

From (10) it follows that the eigenvalue Λ corresponding to $\mathbf{x}$ satisfies the symmetry relation

$$\Lambda \equiv \Lambda(a, b \mid c, d) = (-1)^\nu \Lambda(c, d \mid a, b) \tag{12}$$

If we know the eigenvalues of $\mathbf{T}$ in the regime (5), then we can obtain the eigenvalues in any of the other three ordered states by using (7) and/or (12). Thus there is no less of generality in restricting attention to the regime (5), and we shall do this in the rest of this paper. We refer to this regime (with a, b, c, d all positive) as the *principal regime* (P.R.).

Note that in the P.R. we expect the system to be in an ordered state of antiferroelectric type, this state being a natural generalization of the ordered state of the ice-type F-model.[3] Thus we expect Λ_0 to be positive, Λ_1 to be negative.

3. FUNCTIONAL EQUATIONS FOR EIGENVALUES

The equations that determine the eigenvalues are given in (4.2) and (6.2)–(6.12) of Ref. 1. We make one small change and define k, η, v, and ρ in terms of a, b, c, and d by

$$\begin{aligned} a &= \rho\Theta(2\eta)\Theta(\eta - v)H(\eta + v), \qquad & b &= \rho\Theta(2\eta)H(\eta - v)\Theta(\eta + v) \\ c &= \rho H(2\eta)\Theta(\eta - v)\Theta(\eta + v), \qquad & d &= -\rho H(2\eta)H(\eta - v)H(\eta + v) \end{aligned} \tag{13}$$

where $H(u)$ and $\Theta(u)$ are the elliptic theta functions of argument u and modulus k (§8.192 of Ref. 10). In appendix A we show how k, η, v, and ρ may systematically be calculated from (13).

Since $H(u)$ is an odd function and $\Theta(u)$ is even, the only difference between (13) and our previous definition in Eq. (6.2) of Ref. 1 is that the parametrization of b has been negated. This enable us to consider the P.R. directly, without introducing the mathematical inconvenience of elliptic functions of negative modulus. In fact, from Appendix A we see that in the P.R. we can choose k real and η, v, and ρ pure imaginary, satisfying the inequalities

$$0 < k < 1, \quad \mathrm{Im}(\rho) < 0; \qquad |\,\mathrm{Im}(v)| < \mathrm{Im}(\eta) < \tfrac{1}{2}K' \tag{14}$$

where K, K' are the complete elliptic integrals of the first kind of moduli k, $k' = (1 - k^2)^{1/2}$, respectively (§§110–8.112 of Ref. 10).

We can replace k, η, v by the scaled parameters

$$\tau = \pi K'/2K, \qquad \lambda = -i\pi\eta/K, \qquad \alpha = -i\pi v/K \tag{15}$$

and regard a, b, c, and d as functions of τ, λ, α, and ρ. In the P.R. we see that τ, λ, and α are real. Also, from (14) they satisfy the inequality

$$|\,\alpha\,| < \lambda < \tau \tag{16}$$

Suppose a, b, c, and d to have been given and τ, λ, α, and ρ evaluated. If we then keep τ, λ, and ρ fixed and allow α to assume any complex value, we can regard a, b, c, and d, and hence **T**, as functions of α, or more conveniently of

$$\phi = i\alpha = \pi v/K \tag{17}$$

Writing such an eigenvalue as $T(\phi)$, we have shown in Ret. 1 that $T(\phi)$ is an entire function and satisfies a functional relation. (We use ϕ as the variable in this paper rather than v.)

The fact that we have now negated the parametrization of b slightly complicates the working of Ref. 1 (which is why we did not do so earlier). However, repeating the working with this modification, or else appealing to symmetry relations, we find that Eq. (4.2) of Ref. 1 becomes

$$(-1)^{\nu'} T(\phi)\, Q(\phi) = g(\phi - i\lambda)\, Q(\phi + 2i\lambda) + g(\phi + i\lambda)\, Q(\phi - 2i\lambda) \tag{18}$$

where $\nu' = 0$ or 1 according to whether the eigenvector of **T** corresponding to the eigenvalue $T(\phi)$ has nonzero elements for configurations in which there are an even or odd number of down arrows on each row of vertical bonds of the lattice. Defining a rationalized elliptic theta function $h(\phi)$ by

$$h(\phi) = H(K\phi/\pi)\, \Theta(K\phi/\pi) \tag{19}$$

the function $g(\phi)$ is given by

$$g(\phi) = [\rho\Theta(0)\, h(\phi)]^N \tag{20}$$

and $Q(\phi)$ is also an entire function. It satisfies the quasiperiodic conditions

$$Q(\phi + 2\pi) = (-1)^{\nu'}\, Q(\phi) \tag{21}$$

$$Q(\phi + 2i\tau) = (-1)^{n+\nu}\, e^{n(\tau - i\phi)}\, Q(\phi) \tag{22}$$

where ν has the same meaning as in (11) (it replaces ν'' of Ref. 1), and

$$n = \tfrac{1}{2}N \tag{23}$$

(We take N to be even.)

These equations are in principle sufficient to determine $T(v)$. The elliptic function $h(\phi)$ satisfies the quasiperiodic conditions

$$h(\phi + 2\pi) = -h(\phi), \qquad h(\phi + 2i\tau) = -e^{\tau - i\phi}\, h(\phi) \tag{24}$$

(§§8.182–8.192 of Ref. 10). It is an entire function and has only simple zeros, occurring at

$$\phi = 2m_1\pi + 2im_2\tau \tag{25}$$

(m_1, m_2 integers).

In an ordered antiferroelectric state we expect the number of down arrows per row to be even if n is even, odd if n is odd. Thus

$$(-1)^{\nu'} = (-1)^n \tag{26}$$

In this case it follows from (21) and (22) that it must be possible to factorize $Q(\phi)$ in the form

$$Q(\phi) = \prod_{j=1}^{n} h(\phi - \phi_j) \tag{27}$$

where

$$\exp\left\{ i \sum_{j=1}^{n} \phi_j \right\} = (-1)^{\nu} \tag{28}$$

[To prove this, define $\phi_1, \ldots, \phi_n$ to be the zeros of $Q(\phi)$ in some period rectangle, take the ratio of the two sides of (27), and show that this ratio is a doubly periodic entire function. It is therefore bounded, and hence by Liouville's theorem must be a constant. We can choose this constant to be unity.]

Since $T(\phi)$ is entire, it follows that on setting $\phi = \phi_1, \ldots, \phi_n$ in (18), the l.h.s. vanishes. Equating the r.h.s. to zero, we get n equations for the n unknowns $\phi_1, \ldots, \phi_n$. In principle these can be solved (there will be many solutions, corresponding to the different eigenvalues). We can then evaluate $Q(\phi)$ and $T(\phi)$ for all complex ϕ from (27) and (18). In particular we can evaluate $T(i\alpha)$, where α is real and satisfies (16). Thus we can obtain the eigenvalues in the P.R.

It is of course one thing to note that there are n equations for n unknowns and quite another to solve them. The rest of this paper is concerned with manipulating these equations.

4. INTEGRAL EQUATIONS

The possible locations of $\phi_1, \ldots, \phi_n$ for various eigenvalues have already been discussed in connection with the partition function of the XYZ model[6–8] In this paper we consider only the two maximum eigenvalues and assume (as we did in Ref. 1) that $\phi_1, \ldots, \phi_n$ lie on the real axis. We find that assumption is internally self-consistent and leads to integral equations for the two eigenvalues.

In Appendix B we consider the case when $c \gg a + b + d$, so the system is effectively in a purely ordered state. In this case we are able to verify explicitly that $\phi_1, \ldots, \phi_n$ lie on the real axis and to locate the zeros of $T(\phi)$. We also note that the ratio of the two terms on the r.h.s. of (18), namely

$$r(\phi) = g(\phi - i\lambda)\, Q(\phi + 2i\lambda)/g(\phi + i\lambda)\, Q(\phi - 2i\lambda) \tag{29}$$

is exponentially small (when N is large) in the domain $0 < \operatorname{Im}(\phi) < \min(2\lambda, \tau)$, while it is exponentially large in $\max(-2\lambda, -\tau) < \operatorname{Im}(\phi) < 0$.

This leads us to make the following assumptions, for any a, b, c, and d in the P.R. and for any N.

Assumption A: $\phi_1, \ldots, \phi_n$ lie on the real axis.

Assumption B: $\exists$ a real parameter S_N such that

$$\begin{aligned} &0 < S_N < \min(2\lambda, \tau) \\ &S_N \to \min(2\lambda, \tau) \quad \text{as} \quad N \to \infty \\ &|r(\phi)| < 1 \quad \text{when} \quad 0 < \operatorname{Im}(\phi) < S_N \\ &|r(\phi)| > 1 \quad \text{when} \quad -S_N < \operatorname{Im}(\phi) < 0 \end{aligned}$$

Assumption C: $T(\phi)$ has no zeros on the real axis.

In Appendix C we use these assumptions to perform Wiener–Hopf factorizations[11,12] of $1 + r(\phi)$ and $1 + r^{-1}(\phi)$. This leads us to define the following six functions in terms of c, λ, τ, and ϕ:

$$\ln t(\phi) = \ln c + 2 \sum_{m=1}^{\infty} \frac{\{\sinh^2[(\tau - \lambda)m]\}[\cosh(m\lambda) - \cos(m\phi)]}{m \sinh(2m\tau)\cosh(m\lambda)}, \qquad |\operatorname{Im}(\phi)| < \lambda \tag{30a}$$

$$\mathrm{Dn}(\phi) = 1 + 2 \sum_{m=1}^{\infty} [\cos(m\phi)]/\cosh(m\lambda), \qquad |\operatorname{Im}(\phi)| < \lambda \tag{30b}$$

$$p(\phi) = z^n \prod_{m=0}^{\infty} \{(1 - x^{4m+3}z)(1 - x^{4m+1}z^{-1})/(1 - x^{4m+1}z)(1 - x^{4m+3}z^{-1})\}^N \qquad (\text{all } \phi) \tag{31}$$

$$X(\phi) = \sum_{m=1}^{\infty} \frac{\sinh[m(\tau - 2\lambda)]\cos(m\phi)}{\sinh[m(\tau - \lambda)]\cosh(m\lambda)}, \qquad |\operatorname{Im}(\phi)| < 2\min(\lambda, \tau - \lambda) \tag{32}$$

$$Y(\phi) = \sum_{m=1}^{\infty} \sinh(m\lambda)\cos(m\phi)/\sinh[m(\tau - \lambda)], \qquad \tau > 2\lambda \text{ and } |\operatorname{Im}(\phi)| < \tau - 2\lambda \tag{33}$$

$$Z(\phi) = \tfrac{1}{2}\{\mathrm{Dn}(\phi + i\tau - i\lambda) + \mathrm{Dn}(\phi - i\tau + i\lambda)\}, \qquad \tau < 2\lambda \text{ and } |\operatorname{Im}(\phi)| < 2\lambda - \tau \tag{34}$$

In (29) c is to be regarded as a function of ϕ, defined by (13) and (17). The parameters x and z in (31) are given by

$$x = e^{-\lambda}, \qquad z = e^{i\phi} \tag{35}$$

With these definitions, we find in Appendix C that

$$T(-\phi) = T(\phi), \qquad r(-\phi) = 1/r(\phi) \tag{36}$$

and that $T(\phi)$ is given in terms of $r(\phi)$ by

$$\ln[(-1)^\nu T(\phi)/t^N(\phi)] = (4\pi)^{-1} \int_C \{\ln[1 + r(\phi')]\} \times \{\mathrm{Dn}(\phi' - \phi - i\lambda) + \mathrm{Dn}(\phi' + \phi - i\lambda)\}\, d\phi' \tag{37}$$

provided $|\operatorname{Im}(\phi)| < \lambda$, where C is the straight-line segment $(i\lambda - \pi, i\lambda + \pi)$ in the complex ϕ' plane.

Note from (16) and (17) that in the P.R., $\phi = i\alpha$, where α is real and $|\alpha| < \lambda$. Thus (37) gives the eigenvalue $T(\phi)$ throughout the P.R.

We also find that the function $r(\phi)$ satisfies a nonlinear integral equation.

This equation has a different form according to whether $\lambda \gtrless \frac{1}{2}\tau$ and to whether $|\operatorname{Im}(\phi)| \gtrless 2\min(\lambda, \tau - \lambda)$. We list these forms below. In each there is an integration over a contour C in the complex ϕ' plane, where C is the straight-line segment $(iR - \pi, iR + \pi)$. Note that R need not be the same in each of the three forms, but must be real and satisfy

$$0 < R < S_N \tag{38}$$

as well as the restrictions given after each form.

(i) $|\operatorname{Im}(\phi)| < 2\min(\lambda, \tau - \lambda)$:

$$\ln[(-1)^{n+\nu} r(\phi)/p(\phi)]$$
$$= (2\pi)^{-1} \int_C \{\ln[1 + r(\phi')]\}\{X(\phi - \phi') - X(\phi + \phi')\}\, d\phi' \tag{39a}$$

$$R < 2\min(\lambda, \tau - \lambda) - |\operatorname{Im}(\phi)|$$

(ii) $\lambda < \frac{1}{2}\tau,\ 2\lambda < \operatorname{Im}(\phi) < 2(\tau - \lambda)$:

$$\ln r(\phi) = \pi^{-1} \int_C \{\ln[1 + r(\phi')]\}$$
$$\times \{Y(\phi + \phi' - i\tau) - Y(\phi - \phi' - i\tau)\}\, d\phi' \tag{39b}$$

$$2\lambda + R < \operatorname{Im}(\phi) < 2(\tau - \lambda) - R$$

(iii) $\lambda > \frac{1}{2}\tau,\ 2(\tau - \lambda) < \operatorname{Im}(\phi) < 2\lambda$:

$$\ln\{r(\phi)/[p(\phi)p(\phi - 2i\tau)]\}$$
$$= (2\pi)^{-1} \int_C \{\ln[1 + r(\phi')]\}$$
$$\times \{Z(\phi - \phi' - i\tau) - Z(\phi + \phi' - i\tau)\}\, d\phi' \tag{39c}$$

$$2(\tau - \lambda) + R < \operatorname{Im}(\phi) < 2\lambda - R$$

We show the three regions of applicability in Fig. 1. From the quasi-periodic conditions (22) and (24) and the definitions (20) and (29) we can verify that

$$r(\phi + 2i\tau) = r(\phi) \tag{40}$$

Thus Eqs. (39a)–(39c) are sufficient to determine $r(\phi)$ throughout the complex ϕ plane.

We assume that S_N can be chosen greater than λ (this is certainly true for sufficiently large N, and is probably true for $N \geqslant 2$). Then from Assumption B we see that $|r(\phi')| < 1$ on the r.h.s. of each of Eqs. (37) and (39a)–(39c).

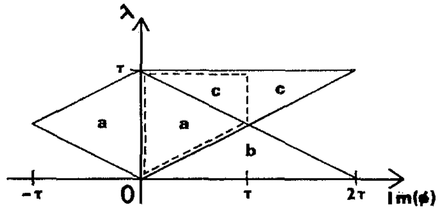

Fig. 1. Regions of the [Im(ϕ), λ] plane within which the forms a, b, and c of (39a)–(39c) and (41) apply. The broken line is the boundary of the region within which $r(\phi)$ is exponentially small when N is large.

The branch of the logarithm on the r.h.s. must be chosen in the obvious way so that $\ln(1 + r)$ is continuous for $| r | < 1$ and vanishes when $r = 0$.

Note that we have in fact two sets of integral equations, depending on whether $\nu = 0$ or 1, i.e., whether the corresponding eigenvector of **T** is symmetric or antisymmetric with respect to reversing all vertical arrows on a row of the lattice. We expect $\nu = 0$ to give the maximum, positive, eigenvalue $\Lambda_0 = T(i\alpha)$, and $\nu = 1$ to give the next largest, negative, eigenvalue Λ_1.

5. THE LIMIT $N \to \infty$

When ϕ' lies on the line segments C used in (37) and (39a)–(39c) we expect $r(\phi')$ to tend to zero exponentially with increasing N. Hence in the limit of N large the equations (39a)–(39c) become

$$r(\phi) \sim (-1)^{n+\nu} p(\phi), \qquad | \operatorname{Im}(\phi)| < 2 \min(\lambda, \tau - \lambda) \tag{41a}$$

$$\sim 1, \qquad 2\lambda < \operatorname{Im}(\phi) < 2(\tau - \lambda) \tag{41b}$$

$$\sim p(\phi)\, p(\phi - 2i\tau), \qquad 2(\tau - \lambda) < \operatorname{Im}(\phi) < 2\lambda \tag{41c}$$

The regions of applicability of these three limiting forms are shown in Fig. 1.

The function $p(\phi)$ is defined by (31). Taking the logarithm of the r.h.s. of (31), Taylor expanding, and rearranging terms, we find that

$$-n^{-1} i \ln p(\phi) = \phi + 2 \sum_{m=1}^{\infty} [\sin(m\phi)]/[m \cosh(m\lambda)] \tag{42a}$$

provided $| \operatorname{Im}(\phi)| < \lambda$. Comparing this with §8.146.4 of Ref. 10, we see that

$$p(\phi) = \exp\{in \operatorname{Am}(\phi)\} \tag{42b}$$

where $\text{Am}(\phi)$ is a rationalized elliptic amplitude function. It satisfies the relations

$$\text{Am}(\phi + 2i\lambda) = \text{Am}(-\phi) = -\text{Am}(\phi) \tag{43a}$$

$$\text{Im Am}(\phi) > 0 \quad \text{when} \quad 0 < \text{Im}(\phi) < 2\lambda \tag{43b}$$

$$\text{Im}[\text{Am}(\phi) + \text{Am}(\phi')] > 0 \quad \text{if} \quad \text{Re}(\phi - \phi') = 0$$

and

$$|\,\text{Im}(\phi) - \lambda\,| + |\,\text{Im}(\phi') + 3\lambda\,| < 2\lambda \tag{43c}$$

Using these properties in (42b) and (41a)–(41c), we find that $r(\phi)$ tends exponentially to zero with increasing N provided $0 < \text{Im}(\phi) < \min(2\lambda, \tau)$. This verifies Assumption B, and the stronger assumption that we made at the beginning of this section.

Further, from (18), (27), and (29), $\phi_1 ,..., \phi_n$ are zeros of $1 + r(\phi)$. We have assumed these to be real. However, for large N and real ϕ, $r(\phi)$ is given by (41a), so to this order $\phi_1 ,..., \phi_n$ are the roots of the equation

$$1 + (-1)^{n+\nu} \exp[in\,\text{Am}(\phi)] = 0 \tag{44}$$

As ϕ increases from 0 to 2π, $\text{Am}(\phi)$ also increases monotonically from 0 to 2π. Also, $\text{Am}(\phi + 2\pi) = \text{Am}(\phi) + 2\pi$. Thus the modulus 2π there are precisely n real roots of (44), namely $\phi_1 ,..., \phi_n$. This verifies (to this order) Assumption A.

From (18) and (29), the zeros of $T(\phi)$ are the zeros of $1 + r(\phi)$ that are not zeros of $Q(\phi)$. Since we have found only n zeros of $1 + r(\phi)$ on the real axis, it follows that $T(\phi)$ has no real zeros. This agrees with Assumption C.

We see therefore that our results are in agreement with our original assumptions and we have found two solutions (taking $\nu = 0$ or 1) to the functional equation (18). From Appendix B we expect these to correspond to the asymptotically degenerate maximum eigenvalues of **T** in the P.R.

5.1. Free Energy

In the limit of N large the r.h.s. of (37) tends exponentially to zero and we see that

$$T(\phi) = (-1)^\nu t^N(\phi) + \text{exponentially smaller terms} \tag{45}$$

In the P.R., $\phi = i\alpha$ and the two largest eigenvalues of **T** are therefore, taking $\nu = 0$ or 1,

$$\Lambda_\nu \equiv T(i\alpha) = (-1)^\nu t^N(i\alpha) + \text{exponentially smaller terms} \tag{46}$$

From (29) we see that $t(i\alpha)$ is positive and to this order Λ_0 and Λ_1 are equal in magnitude but opposite in sign, Λ_0 being positive, as expected.

The free energy f per site of the infinite lattice is given by

$$-\beta f = \lim_{N\to\infty} N^{-1} \ln \Lambda_0 = \ln t(i\alpha) \tag{47}$$

which is the result previously obtained [Eq. (7.7) of Ref. 1].

5.2. Interfacial Tension

From (2) the interfacial tension σ is given by

$$-\beta\sigma = \lim_{N\to\infty} N^{-1} \ln \ln(-\Lambda_0/\Lambda_1) \tag{48}$$

To evaluate this, we must consider the dominant behavior of the r.h.s. of (37) when N is large. When $0 < \lambda < 2\tau/3$ to first order the function $r(\phi')$ on the r.h.s. of (37) is given by (41a). Thus to this order

$$\ln(-\Lambda_0/\Lambda_1) = (-1)^n(2\pi)^{-1} \int_{-\pi}^{\pi} p(i\lambda + u)[\mathrm{Dn}(u - i\alpha) + \mathrm{Dn}(u + i\alpha)]\, du \tag{49}$$

where we have replaced ϕ by $i\alpha$ and the integration variable ϕ' by $i\lambda + u$.

Replacing ϕ in the definition (31) of $p(\phi)$ by $i\lambda + u$ and comparing the resulting infinite product with §8.146.23 of Ref. 10,[3] we see that

$$p(i\lambda + u) = (-1)^n k_2{}^n sn^N(K_2 u/\pi, k_2) \tag{50}$$

where k_2 is the elliptic modulus with associated elliptic integrals K_2, K_2' such that

$$\pi K_2'/K_2 = 2\lambda \tag{51}$$

The function $p(i\lambda + u)$ therefore has a maximum at $u = \pm\pi$, its value being $(-1)^n k_2{}^n$. Performing an integration by steepest descents in (49), we see that when N is large

$$\ln(-\Lambda_0/\Lambda_1) \sim (k_2'K_2)^{-1}(2\pi/N)^{1/2}\, k_2^{N/2}\, \mathrm{Dn}(\pi + i\alpha) \tag{52}$$

Substituting this result into (48), we obtain

$$\beta\sigma = -\tfrac{1}{2} \ln k_2 \tag{53}$$

[3] The third printing, 1967, of Ref. 10 contains an error in §8.146.23. The square root of q should be replaced by $q^{1/4}$.

When $2\tau/3 < \lambda < \tau$, $r(\phi')$ in (37) is given by (39c) rather than (39a). Thus to first order we must substitute $r(\phi') = p(\phi')\,p(\phi' - 2i\tau)$ in (37). However, the r.h.s. is then independent of ν and to this order $-\Lambda_0/\Lambda_1 = 1$. We therefore have to go to higher orders to calculate σ. We do this in Appendix D and show that we again obtain (49) and (53). Thus our results are valid throughout the P.R.

Note that $0 < k_2 < 1$, so σ is positive, as expected.

6. BEHAVIOR NEAR THE CRITICAL TEMPERATURE

The Boltzmann weights a, b, c, and d are functions[1] of the temperature T. In the P.R., c decreases as T increases from zero until at the critical temperature T_c the Boltzmann weights satisfy the relation $c = a + b + d$.

We have discussed the behavior of the free energy f in Ref. 1 and shown that near T_c the dominant singular contribution to f is proportional to

$$[\cot(\tfrac{1}{2}\pi^2/\bar{\mu})]\ |\ T - T_c\ |^{\pi/\bar{\mu}} \tag{54}$$

or if $\pi/2\bar{\mu} = m =$ integer, to

$$2\pi^{-1}(T - T_c)^{2m} \ln |\ T - T_c\ | \tag{55}$$

To avoid confusion with the standard notation for the critical exponent of the interfacial tension, we have replaced the μ of Ref. 1 by $\bar{\mu}$. It is defined for all temperatures by

$$\bar{\mu} = \pi\lambda/\tau \tag{56}$$

and at T_c is given explicitly in terms of a, b, c, and d by $0 < \bar{\mu} < \pi$ and

$$\begin{aligned} \cos\bar{\mu} &= (cd - ab)/(cd + ab) \quad &&\text{if} \quad a = b + c + d \quad \text{or} \quad b = a + c + d \\ &= (ab - cd)/(ab + cd) \quad &&\text{if} \quad c = a + b + d \quad \text{or} \quad d = a + b + c \end{aligned} \tag{57}$$

When T increases from zero to T_c, k_2 increases from zero to one, so the interfacial tension σ decreases from infinity to zero. To obtain the behavior of σ near T_c, we note from §8.113 of Ref. 10 that near $k_2 = 1$

$$\ln(4/k_2') \simeq \pi K_2/2K_2' \tag{58}$$

where $k_2' = (1 - k_2^2)^{1/2}$. Hence

$$\ln k_2 \simeq -8\exp(-\pi K_2/K_2') = -8\exp(-\pi^2/2\lambda) \tag{59}$$

using (51).

From (56) and Eq. (D4) of Ref. 1, we see that

$$\lambda = \bar{\mu} K_l / K_l' \tag{60}$$

Hence from (53) and (59)

$$\sigma \simeq 4 q_l^{\pi/(2\bar{\mu})} \tag{61}$$

where q_l is the q defined in Eq. (E7) of Ref. 1. We have shown there that q_l has a simple zero at $T = T_c$, so near the critical temperature

$$\sigma \propto (T_c - T)^{\pi/(2\bar{\mu})} \tag{62}$$

When $ab = cd$ the zero-field eight-vertex model decouples into two independent Ising models. In this case $\bar{\mu} = \pi/2$ and we regain the known critical behavior of the Ising model,[13,14] namely a logarithmic divergence of the specific heat and a linear vanishing of σ.

Note also that for the Ising model case $\lambda = \frac{1}{2}\tau$ (at all temperatures), so the function $X(\phi)$ defined by (32) vanishes identically. From (39a) it follows that $r(\phi) = (-1)^{n+\nu} p(\phi)$ exactly. The two maximum eigenvalues can then be obtained exactly for all N from (37).

7. SCALING PREDICTIONS

Using the usual notations of scaling theory, near T_c the free energy f is expected to have a dominant singularity proportional to

$$(T - T_c)^{2-\alpha} \qquad \text{or} \qquad (T_c - T)^{2-\alpha'} \tag{63}$$

while the interfacial tension σ is expected to be proportional to

$$(T_c - T)^{\mu} \tag{64}$$

We see that our results (54) and (62) are in agreement with these predictions, with

$$\alpha = \alpha' = 2 - \pi/\bar{\mu} \tag{65}$$

$$\mu = \pi/2\bar{\mu} \tag{66}$$

In addition, the critical exponent ν of the correlation length has recently been calculated by Johnson *et al.*[15] and found to be given by

$$\nu = \pi/2\bar{\mu} \tag{67}$$

Scaling theory predicts that these critical exponents should satisfy the relations [Eqs. (12) and (13) of Ref. 18, (28) of Ref. 17, (9.3.15) of Ref. 18]:

$$\mu + \nu = 2\beta + \gamma' = 2 - \alpha' = 2 - \alpha \tag{68a}$$

$$\mu = (d - 1)\nu \tag{68b}$$

where d is the dimensionality of the system. In our case $d = 2$ and we see that the exact results (65)–(67) for the eight-vertex model do indeed satisfy the scaling predictions (68a) and (68b).

APPENDIX A. PARAMETRIZATION OF THE BOLTZMANN WEIGHTS

To obtain k, η, v, and ρ systematically from the four equations (13), first eliminate ρ by taking the ratios of a, b, c, and d. From §8.191.1 of Ref. 10 this gives

$$a : b : c : d = \operatorname{sn}(\eta + v) : \operatorname{sn}(\eta - v) : \operatorname{sn}(2\eta) : -k \operatorname{sn}(2\eta) \operatorname{sn}(\eta + v) \operatorname{sn}(\eta - v) \tag{A1}$$

To eliminate v, we note from (A1) that

$$cd/ab = -k \operatorname{sn}^2(2\eta) \tag{A2}$$

We can also prove that

$$(c^2 + d^2 - a^2 - b^2)/2ab = \operatorname{cn}(2\eta) \operatorname{dn}(2\eta) \tag{A3}$$

[One way to prove (A3) is to use the formulas:

$$\operatorname{sn}^2(2\eta) - \operatorname{sn}^2(\eta + v) = [1 - k^2 \operatorname{sn}^2(2\eta) \operatorname{sn}^2(\eta + v)] \operatorname{sn}(\eta - v) \operatorname{sn}(3\eta + v) \tag{A4}$$

$$\operatorname{sn}(3\eta + v) - \operatorname{sn}(\eta - v) = 2 \operatorname{sn}(\eta + v) \operatorname{cn}(2\eta) \operatorname{dn}(2\eta)/[1 - k^2 \operatorname{sn}^2(2\eta) \operatorname{sn}^2(\eta + v)]$$

These formulas can be deduced from §§8.156.1, 8.154.4, and 5 of Ref. 10.]

We can use §§8.154.4 and 5 of Ref. 10 to eliminate η from (A2) and (A3). This gives

$$k + k^{-1} - 2 = (a - b + c + d)(a + b - c + d)(a + b + c - d)(a - b - c - d)/4abcd \tag{A5}$$

In an ordered state one of the weights a, b, c, or d is greater than the sum of the other three (and all are positive). From (A5) it follows that

$$k + k^{-1} > 2 \tag{A6}$$

and hence we can choose k so that

$$0 < k < 1 \tag{A7}$$

In the P.R. $c > a + b + d$, so the l.h.s. of (A3) is greater than one. The l.h.s. of (A2) is positive. It follows that we can choose η to be pure imaginary and such that

$$0 < \mathrm{Im}(\eta) < \tfrac{1}{2}K' \tag{A8}$$

The ratios a/c and b/c are positive and less than one. From (A1) it follows that we can choose v to be pure imaginary and such that

$$|\,\mathrm{Im}(v)| < \mathrm{Im}(\eta) \tag{A9}$$

From §§8.192 and 8.181 of Ref. 10 we see that all the elliptic Θ functions in (13) are positive (real), while the H functions are positive imaginary. Thus ρ must be negative imaginary.

This completes the proof of the inequalities (14).

APPENDIX B. THE PURELY ORDERED LIMIT

We consider the low-temperature limit when $c \gg a + b + d$. In this case we see from (A5) that k is small. Hence $K \simeq \frac{1}{2}\pi$, K' and τ are large, and we can use the approximate formulas (§§8.197.3, 8.192, 8.181, and 8.191 of Ref. 10):

$$k \simeq 4e^{-\tau} \tag{B1}$$

$$\Theta(u) \simeq 1, \qquad k^{-1/2}H(u) \simeq \mathrm{sn}\, u \simeq \sin u \tag{B2}$$

provided $K' - |\,\mathrm{Im}(u)| \gg 1$.

From (A2) and (15) it follows that $\mathrm{Im}(\eta) \simeq \frac{1}{2}\lambda$ is large, but that λ/τ is of order unity and less than one. Thus from (13)

$$c \sim i\rho \exp(\lambda - \tfrac{1}{2}\tau) \tag{B3}$$

provided $\lambda - |\,\mathrm{Im}(\phi)| \gg 1$.

From (19) the function $h(\phi)$ is given by

$$h(\phi) \simeq 2e^{-\tau/2} \sin(\tfrac{1}{2}\phi) \tag{B4}$$

provided $2\tau - |\,\mathrm{Im}(\phi)| \gg 1$.

Substituting (B4) into the expressions (20) and (27) for $g(\phi)$ and $Q(\phi)$, assuming that $\phi_1, \ldots, \phi_n$ are real, and substituting these in turn into (18), Eq. (18) becomes

$$(-1)^{\nu'} c^{-N} T(\phi) \prod_{j=1}^{n} (1 - z/z_j) \simeq z^n + (-1)^n/(z_1 \cdots z_n) \tag{B5}$$

where

$$z = e^{i\phi}, \qquad z_j = e^{i\phi_j} \tag{B6}$$

and provided that $\min(\lambda, 2\tau - 2\lambda) - |\operatorname{Im}(\phi)| \gg 1$.

This last condition is satisfied if ϕ is real, and the l.h.s. of (B5) vanishes if $\phi = \phi_1, \ldots, \phi_n$. Thus to this approximation $z_1, \ldots, z_n$ are the distinct roots of the equation

$$z^n + (-1)^n/(z_1 \cdots z_n) = 0 \tag{B7}$$

and hence

$$(z_1 \cdots z_n)^2 = 1 \tag{B8}$$

Note that from (28) and (B6) we have the exact equation (for all values of a, b, c, and d in the P.R.)

$$z_1 \cdots z_n = (-1)^{\nu} \tag{B9}$$

This agrees with (B8). There are two solutions for $z_1, \ldots, z_n$, depending on whether we take the positive or negative root in (B8), i.e., whether $\nu = 0$ or 1 in (B9). In either case we obtain

$$\prod_{j=1}^{n} (1 - z/z_j) \simeq 1 + (-1)^n (z_1 \cdots z_n)\, z^n \tag{B10}$$

Substituting this back into (B5), the polynomials on both sides cancel, leaving [using (26)]

$$T(\phi) \simeq (-1)^{\nu} c^N, \qquad \nu = 0 \quad \text{or} \quad 1 \tag{B11}$$

provided $\min(\lambda, 2\tau - 2\lambda) - |\operatorname{Im}(\phi)| \gg 1$.

We see that these are indeed the asymptotically degenerate maximum eigenvalues in the purely ordered limit. Also, from (B6), (B7), and (B9), $z_1, \ldots, z_n$ all lie on the unit circle, so $\phi_1, \ldots, \phi_n$ are real, as assumed.

Substituting (B4) into (20), (27), and (29), we can obtain $r(\phi)$, the ratio of the two terms on the r.h.s. of (18), in this limit. There are different cases to consider, depending on the values of $\operatorname{Im}(\phi)$ and λ/τ. We show some of the regions of the $[\operatorname{Im}(\phi), \lambda]$ plane that we are led to consider in Fig. 2. In the

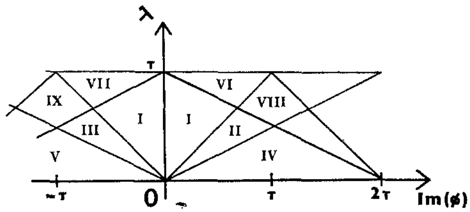

Fig. 2. The regions I–IX of the [Im(ϕ), λ] plane referred to in Eq. (B12).

interior of these regions I–IX (i.e., a distance large compared with unity from the boundaries) we find the following results:

$$\begin{aligned} \text{I:} \quad & r(\phi) \sim (-1)^{n+\nu} z^n \\ \text{II:} \quad & \sim (-1)^{n+\nu} x^{2n} z^{-n} \\ \text{III:} \quad & \sim (-1)^{n+\nu} x^{-2n} z^n \\ \text{IV, V:} \quad & \sim 1 \\ \text{VI, VII:} \quad & \sim q^n x^{-2n} \\ \text{VIII:} \quad & \sim q^n z^{-2n} \\ \text{IX:} \quad & \sim q^{-n} z^{-2n} \end{aligned} \tag{B12}$$

where z is defined by (B6) and

$$x = e^{-\lambda}, \qquad q = e^{-2\tau} \tag{B13}$$

Examining these forms, we see that in this limit $r(\phi)$ is exponentially small [i.e., $N^{-1} \ln |r(\phi)|$ tends to a negative limit as $N \to \infty$] when $0 < \text{Im}(\phi) < \min(2\lambda, \tau)$. It is exponentially large when $-\min(2\lambda, \tau) < \text{Im}(\phi) < 0$.

From (18) and (29) the zeros of $T(\phi)$ are also zeros of $1 + r(\phi)$. From (B3) and (B11) we see that $T(\phi)$ has no zeros on the real axis. Looking again at the forms (B12), we see that its zeros must lie (to modulus $2i\tau$) close to the lines $\text{Im}(\phi) = \pm\min(2\lambda, \tau)$.

We have derived these results for the purely ordered limit. However, we do not expect a qualitative change as we raise the temperature and reduce c to values comparable with a, b, and d (so long as we stay in the P.R., $c > a + b + d$). In particular, we still expect $r(\phi)$ to be exponentially small in some domain above the real axis, and large in a domain below it. Thus in these domains one of the terms on the r.h.s. of (18) is much less than the other when N is large. This is the key property that we exploit in this paper to obtain a reformulation of (18) that is appropriate for examining the behavior as $N \to \infty$.

APPENDIX C

Wiener–Hopf Factorizations of $1 + [r(\phi)]^{\pm 1}$

From (18), (26), and (29) we see that

$$1 + r(\phi) = (-1)^n T(\phi)\, Q(\phi)/[g(\phi + i\lambda)\, Q(\phi - 2i\lambda)] \qquad \text{(C1)}$$

By Assumption B, $|r(\phi)| < 1$ when $0 < \mathrm{Im}(\phi) < S_N$. In fact from Appendix B we expect $r(\phi)$ to tend to zero exponentially with increasing N in this domain. This suggests that we may usefully use the Wiener–Hopf technique[11,12] and define two functions $P_1(\phi)$ and $P_2(\phi)$ by

$$\ln P_1(\phi) = -\int_{C_1} \{\ln[1 + r(\phi')]\} U(\phi - \phi')\, d\phi', \qquad \mathrm{Im}(\phi) < R_1 \quad \text{(C2)}$$

$$\ln P_2(\phi) = \int_{C_2} \{\ln[1 + r(\phi')]\} U(\phi - \phi')\, d\phi', \qquad \mathrm{Im}(\phi) > R_2 \qquad \text{(C3)}$$

where

$$U(\phi) = (2\pi)^{-1}(1 - e^{i\phi})^{-1} \qquad \text{(C4)}$$

In the above equations C_1 and C_2 are the straightline segments $(iR_1 - \pi, iR_1 + \pi)$ and $(iR_2 - \pi, iR_2 + \pi)$, respectively, in the complex ϕ' plane. The parameters R_1 and R_2 are real and

$$0 < R_i < S_N, \qquad i = 1 \quad \text{or} \quad 2 \qquad \text{(C5)}$$

From Assumption B it follows that $|r(\phi')| < 1$ on C_1 and C_2. Hence the logarithms on the r.h.s. of (C2) and (C3) are analytic, single-valued functions of ϕ'. From (20), (21), (24), (27), and (29) they are also periodic, of period 2π. The branch of the logarithm is to be chosen so as to be a continuous function of r, vanishing when $r = 0$.

If $R_2 < \mathrm{Im}(\phi) < R_1$, then adding (C2) and (C3) and using $r(\phi' + 2\pi) = r(\phi')$, the r.h.s. can be evaluated by Cauchy's residue theorem, giving

$$P_1(\phi)\, P_2(\phi) = 1 + r(\phi) \qquad \text{(C6)}$$

From its definition (C2), we see that $P_1(\phi)$ is an analytic nonzero function of ϕ for $\mathrm{Im}(\phi) < R_1$, while $P_2(\phi)$ is analytic and nonzero for $\mathrm{Im}(\phi) > R_2$. When $\mathrm{Im}(\phi) > R_1$, $P_1(\phi)$ is to be regarded as defined by (C6), while $P_2(\phi)$ is defined by (C6) when $\mathrm{Im}(\phi) < R_2$.

The functions $T(\phi)$, $Q(\phi)$, and $g(\phi)$ are entire, so from (C1), $1 + r(\phi)$ is a meromorphic function. From (C5) it follows that $P_1(\phi)$ and $P_2(\phi)$ are also meromorphic functions, the zeros and poles of $P_1(\phi)$ $[P_2(\phi)]$ being those of $1 + r(\phi)$ above C_1 [below C_2].

We can obtain infinite product expansions of $g(\phi)$ and $Q(\phi)$ that exhibit their zeros explicitly. To do this we use (20), (22), (28), and the formula [Eq. (3.7) and §§8.181 and 8.192 of Ref. 10]

$$h(\phi) = 2\gamma^2 q^{1/4} \sin(\tfrac{1}{2}\phi) \prod_{m=1}^{\infty} [(1 - q^m e^{i\phi})(1 - q^m e^{-i\phi})] \tag{C7}$$

where

$$q = e^{-2\tau} \tag{C8}$$

$$\gamma = \prod_{m=1}^{\infty} (1 - q^{2m}) \tag{C9}$$

This gives

$$g(\phi) = x^N \xi^N e^{-in\phi} A(\phi)\, A(2i\tau - \phi) \tag{C10a}$$

$$= x^N \xi^N e^{in\phi} A(\phi + 2i\tau)\, A(-\phi) \tag{C10b}$$

and, to within a constant normalization factor that cancels out of (18) and (29)

$$Q(\phi) = e^{-in\phi/2} F(\phi)\, G(\phi - 2i\tau) \tag{C11a}$$

$$= (-1)^{n+\nu}\, e^{in\phi/2} F(\phi + 2i\tau)\, G(\phi) \tag{C11b}$$

where

$$\xi = [i\rho\Theta(0)\, \gamma^2 q^{1/4} x^{-1}]^N \tag{C12}$$

$$A(\phi) = \prod_{m=0}^{\infty} (1 - q^m e^{i\phi})^N \tag{C13}$$

$$F(\phi) = \prod_{j=1}^{n} \prod_{m=0}^{\infty} [1 - q^m e^{i(\phi-\phi_j)}] \tag{C14}$$

$$G(\phi) = \prod_{j=1}^{n} \prod_{m=0}^{\infty} [1 - q^m e^{-i(\phi-\phi_j)}] \tag{C15}$$

The parameter x need not be defined at this stage, but for convenience in the next equation we set

$$x = e^{-\lambda} \tag{C16}$$

The functions $A(\phi)$, $F(\phi)$, and $G(\phi)$ are entire. Using Assumption A, $F(\phi)$ and $A(\phi)$ are nonzero in the upper halfplane, and $G(\phi)$ is nonzero in the lower half-plane.

Substituting the forms (C11a), (C10a), and (C11b) of $Q(\phi)$, $g(\phi + i\lambda)$, and $Q(\phi - 2i\lambda)$ into (C1) and (C6), after some cancellations we find that

$$\begin{aligned} P_1(\phi)\, P_2(\phi) \\ &= (-1)^{\nu} \xi^{-N} T(\phi)\, F(\phi)\, G(\phi - 2i\tau) \\ &\quad \times [A(\phi + i\lambda)\, A(2i\tau - i\lambda - \phi)\, F(\phi + 2i\tau - 2i\lambda)\, G(\phi - 2i\lambda)]^{-1} \end{aligned} \tag{C17}$$

Basically what we want to do now is to factorize $T(\phi)$ into two analytic factors $T_+(\phi)$ and $T_-(\phi)$, the former being nonzero in the upper half-plane and the latter nonzero in the lower half-plane. We then want to identify $P_1(\phi)$ $[P_2(\phi)]$ with the product of the factors on the r.h.s. of (C17) that are nonzero in the lower [upper] half-plane.

However, since two functions that have the same poles and zeros are not necessarily identical, we are forced to adopt a slightly roundabout route. We define $T_+(\phi)$ and $T_-(\phi)$ by

$$P_1(\phi) = T_-(\phi)\, G(\phi - 2i\tau)/[A(2i\tau - i\lambda - \phi)\, G(\phi - 2i\lambda)] \tag{C18}$$

$$P_2(\phi) = (-1)^{\nu} \xi^{-N} T_+(\phi)\, F(\phi)/[A(\phi + i\lambda)\, F(\phi + 2i\tau - 2i\lambda)] \tag{C19}$$

then from (C2), (C3), and (C13)–(C15), $T_-(\phi)$ is analytic and nonzero for $\mathrm{Im}(\phi) < S_N$, and $T_+(\phi)$ is analytic and nonzero for $\mathrm{Im}(\phi) > 0$. Also, from these equations we can deduce that

$$\begin{aligned} T_-(\phi) &\to 1 \quad \text{as} \quad \mathrm{Im}(\phi) \to -\infty \\ T_+(\phi) &\to \text{const} \quad \text{as} \quad \mathrm{Im}(\phi) \to +\infty \\ T_{\pm}(\phi + 2\pi) &= T_{\pm}(\phi) \end{aligned} \tag{C20}$$

Further, using (C17), it is apparent that

$$T(\phi) = T_+(\phi)\, T_-(\phi) \tag{C21}$$

From (18) and (29) the zeros of $T(\phi)$ are also zeros of $1 + r(\phi)$. From Assumptions B and C it follows that $T(\phi)$ is nonzero for $-S_N < \mathrm{Im}(\phi) < S_N$. We also know from Ref. 1 that $T(\phi)$ is entire. Using (C21) to define $T_+(\phi)$ in $-S_N < \mathrm{Im}(\phi) \leqslant 0$, it follows from the above remarks that

$$\begin{aligned} &T_+(\phi) \quad \text{is analytic and nonzero for} \quad \mathrm{Im}(\phi) > -S_N \\ &T_-(\phi) \quad \text{is analytic and nonzero for} \quad \mathrm{Im}(\phi) < S_N \end{aligned} \tag{C22}$$

We can now regard (C18) and (C19) as a natural factorization of (C17), $P_1(\phi)$ containing all the poles and zeros of the r.h.s. of (C17) that lie above iS_N, $P_2(\phi)$ all the poles and zeros that lie on or below the real axis.

We now use the assumption that $|r(\phi)| > 1$ when $-S_N < \text{Im}(\phi) < 0$ and define two more functions

$$\ln P_3(\phi) = -\int_{C_3} \{\ln[1 + 1/r(\phi')]\} U(\phi - \phi')\, d\phi', \qquad \text{Im}(\phi) < R_3 \tag{C23}$$

$$\ln P_4(\phi) = \int_{C_4} \{\ln[1 + 1/r(\phi')]\} U(\phi - \phi')\, d\phi', \qquad \text{Im}(\phi) > R_4 \tag{C24}$$

where

$$-S_N < R_{3,4} < 0 \tag{C25}$$

and C_j is the straight-line segment $(iR_j - \pi, iR_j + \pi)$ in the complex ϕ' plane ($j = 3$ or 4).

We apply similar reasoning to that used to obtain (C18)–(C22). First,

$$\begin{aligned} P_3(\phi)\, P_4(\phi) &= 1 + 1/r(\phi') \\ &= (-1)^n T(\phi)\, Q(\phi)[g(\phi - i\lambda)\, Q(\phi + 2i\lambda)]^{-1} \\ &= (-1)^\nu \xi^{-N} T(\phi)\, F(\phi + 2i\tau)\, G(\phi) \\ &\quad \times [A(\phi - i\lambda + 2i\tau)\, A(i\lambda - \phi)\, F(\phi + 2i\lambda)\, G(\phi - 2i\tau + 2i\lambda)]^{-1} \end{aligned} \tag{C26}$$

using (29), (18), (C11b), (C10b), (C11a), and (C16).

We define $T_+(\phi)$ and $T_-(\phi)$ by

$$P_3(\phi) = T_-(\phi)\, G(\phi)/[A(i\lambda - \phi)\, G(\phi - 2i\tau + 2i\lambda)] \tag{C27}$$

$$P_4(\phi) = (-1)^\nu \xi^{-N} T_+(\phi)\, F(\phi + 2i\tau)/[A(\phi - i\lambda + 2i\tau)\, F(\phi + 2i\lambda)] \tag{C28}$$

and verify that they satisfy (C21), (C20), and (C22).

However, the factorization (C21) of $T(\phi)$, subject to the conditions (C20) and (C22), is unique. [To see this, consider another factorization $T(\phi) = T_+'(\phi)\, T_-'(\phi)$. Then from (C22), $T'(\phi)/T(\phi)$ is an entire function of ϕ, while from (C20) it is periodic of period 2π, tends to one as $\text{Im}(\phi) \to -\infty$, and tends to some constant as $\text{Im}(\phi) \to +\infty$. If is therefore bounded, so by Liouville's theorem it must be a constant, namely unity. Thus $T_-'(\phi) = T_-(\phi)$.]

The functions $T_-(\phi)$ defined by (C18) and (C27) are therefore the same; similarly, so are $T_+(\phi)$ defined by (C19) and (C28).

Derivation of the Integral Equations

The functions $A(\phi)$ is known, being given by (C13). The functions $P_j(\phi)$, $j = 1, \ldots, 4$, are "almost known," in the sense that when N is large we expect $r(\phi')$ to be exponentially small in the definitions (C2) and (C3), and exponentially large in (C23) and (C24). Thus in this limit each $P_j(\phi)$ will be identically equal to unity, so long as ϕ lies in the domain specified in each definition.

We can therefore regard (C18), (C19), (C27), and (C29) as four equations for the four unknown functions $F(\phi)$, $G(\phi)$, $T_+(\phi)$, and $T_-(\phi)$. Eliminating $T_+(\phi)$ between (C19) and (C28), we obtain

$$S(\phi) = S(\phi + 2i\tau - 2i\lambda)\, P_2(\phi)/P_4(\phi) \tag{C29}$$

where

$$S(\phi) = F(\phi)\, F(\phi + 2i\lambda)/A(\phi + i\lambda) \tag{C30}$$

Letting $\mathrm{Im}(\phi) \to +\infty$ in these equations, we see from (C13) and (C14) that $F(\phi)$, $A(\phi) \to 1$, so $S(\phi) \to 1$ and $P_2(i\infty) = P_4(i\infty)$. One can therefore replace $P_2(\phi)$ and $P_4(\phi)$ in (C29) by

$$P_j'(\phi) = P_j(\phi)/P_j(i\infty), \qquad j = 2 \quad \text{or} \quad 4 \tag{C31}$$

This avoids convergence problems in the following discussion.

Since, from (16), $\tau > \lambda > 0$, we can solve (C29) by recursion for $S(\phi)$, giving

$$S(\phi) = \prod_{m=0}^{\infty} \{P_2'[\phi + 2im(\tau - \lambda)]/P_4'[\phi + 2im(\tau - \lambda)]\} \tag{C32}$$

Writing (C30) as

$$F(\phi) = A(\phi + i\lambda)\, S(\phi)/F(\phi + 2i\lambda) \tag{C33}$$

and solving this by recursion for $F(\phi)$, we obtain

$$F(\phi) = F_0(\phi) \prod_{m=0}^{\infty} [S(\phi + 4im\lambda)/S(\phi + 2i\lambda + 4im\lambda)] \tag{C34}$$

where $F_0(\phi)$ is a known function, being given by

$$F_0(\phi) = \prod_{m=0}^{\infty} \{A[\phi + i(4m + 1)\lambda]/A[\phi + i(4m + 3)\lambda]\} \tag{C35}$$

Substituting (C32) into (C34), taking logarithms of both sides, and using (C31), (C3), and (C24), we find that

$$\ln[F(\phi)/F_0(\phi)] = \int_{C_2} \{\ln[1 + r(\phi')]\} J(\phi - \phi')\, d\phi' - \int_{C_4} \{\ln[1 + 1/r(\phi')]\} J(\phi - \phi')\, d\phi', \quad \mathrm{Im}(\phi) > R_2 \tag{C36}$$

where

$$J(\phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \{U[\phi + 2im\lambda + 2il(\tau - \lambda)] - U(i\infty)\} \quad \text{(C37)}$$

Using (C4) to expand $U(\phi)$ in increasing powers of $e^{i\phi}$ and substituting this expansion into (C37), the summations over l and m can be performed to give

$$J(\phi) = (2\pi)^{-1} \sum_{j=1}^{\infty} e^{ij\phi}/[(1 + x^{2j})(1 - q^j x^{-2j})] \quad \text{(C38)}$$

$\text{Im}(\phi) > 0$, where q and x are defined by (C8) and (C16).

Equation (C36) enables us to express $F(\phi)$ in terms of the values of $r(\phi')$ on C_2 or C_4. In particular, when N is large we expect the r.h.s. of (C36) to be small, so that

$$F(\phi) \sim F_0(\phi), \qquad \text{Im}(\phi) > 0 \quad \text{(C39)}$$

We can obtain a similar equation for $G(\phi)$ by eliminating $T_-(\phi)$ between (C18) and (C27) and solving the resulting equation recursively. [Only now we iterate toward $-i\infty$ in the ϕ plane and we keep the original normalizations of $P_1(\phi)$ and $P_3(\phi)$.] This gives

$$\begin{aligned} &\ln[G(\phi)/F_0(-\phi)] \\ &\quad = -\int_{C_1} \{\ln[1 + r(\phi')]\} J(\phi' - \phi)\, d\phi' \\ &\qquad + \int_{C_3} \{\ln[1 + 1/r(\phi')]\} J(\phi' - \phi)\, d\phi, \qquad \text{Im}(\phi) < R_3 \quad \text{(C40)} \end{aligned}$$

Hence when N is large

$$G(\phi) \sim F_0(-\phi), \qquad \text{Im}(\phi) < 0 \quad \text{(C41)}$$

We can now substitute the expressions given by (C36) and (C40) for $F(\phi)$ and $G(\phi)$ into (C11) and (29) to obtain an equations for $r(\phi)$. We get different forms (applicable in different domains of the ϕ plane) according to whether we use (C11a) or (C11b) to evaluate $Q(\phi)$. In particular, if we use (C11a) for $Q(\phi + 2i\lambda)$ and (C11b) for $Q(\phi - 2i\lambda)$ in (29), we obtain

$$\begin{aligned} \ln[(-1)^{n+\nu} r(\phi)/p(\phi)] &= (2\pi)^{-1} \int_{C_2} \{\ln[1 + r(\phi')]\} X(\phi - \phi')\, d\phi' \\ &\quad - (2\pi)^{-1} \int_{C_3} \{\ln[1 + 1/r(\phi')]\} X(\phi - \phi')\, d\phi' \quad \text{(C42)} \end{aligned}$$

provided

$$R_2 - 2\min(\lambda, \tau - \lambda) < \mathrm{Im}(\phi) < R_3 + 2\min(\lambda, \tau - \lambda)$$

We have chosen $R_1 = R_2$, $R_4 = R_3$ (in this equation this involves no loss of generality); $p(\phi)$ is given by (31) [we have used (C13)]; and $X(\phi)$ is given by

$$\begin{aligned}(2\pi)^{-1}X(\phi) &= J(\phi + 2i\lambda) + J(2i\lambda - \phi)\\ &\quad - J(\phi + 2i\tau - 2i\lambda) - J(2i\tau - 2i\lambda - \phi)\end{aligned} \tag{C43}$$

Using (C38), (C8), and (C16), we see that this is the same as the definition (32) of $X(\phi)$ in the text.

In the limit of N large we expect the r.h.s. of (C42) to be small (since r and $1/r$ in it are small). Hence in this limit, taking $R_2 \to 0^+$, $R_3 \to 0^-$,

$$r(\phi) \sim (-1)^{n+\nu}p(\phi), \qquad |\mathrm{Im}(\phi)| < 2\min(\lambda, \tau - \lambda) \tag{C44}$$

From (C43) it is apparent that $p(\phi)$ satisfies the symmetry relation

$$p(-\phi) = 1/p(\phi) \tag{C45}$$

and hence so does $r(\phi)$ in this limit. Further, if we suppose that (C42) can be solved recursively for $r(\phi)$, then

$$r(-\phi) = 1/r(\phi) \tag{C46}$$

for all *finite* N. To see this, take $R_3 = -R_2$ in (C42) and replace ϕ' in the second integral by $-\phi'$. Using (C46) in this integral, we obtain Eq. (39a). Since $X(\phi)$ is an even function, it is apparent that the r.h.s. of (39a) is an odd function of ϕ, and hence $r(\phi)$ on the l.h.s. satisfies (C46).

The relation (C46) is simply equivalent to asserting that $\phi_1, \ldots, \phi_n$ occur in pairs $(\phi_j, -\phi_j)$, which we could have assumed for reasons of symmetry in the beginning. However, we feel that the derivation given above makes the reasoning a little clearer.

We have now deduced the form (39a) of the integral equation for $r(\phi)$. To obtain (39b), we use the form (C11a) of $Q(\phi)$ for both $Q(\phi + 2i\lambda)$ and $Q(\phi - 2i\lambda)$ in (29). Substituting the expressions (C36) and (C40) for $F(\phi)$ and $G(\phi)$, choosing $R_1 = R_2 = -R_3 = -R_4 = R$, and using (C46), we obtain (39b) with

$$\begin{aligned}\pi^{-1}Y(\phi) &= J(\phi + i\tau - 2i\lambda) + J(i\tau - 2i\lambda - \phi)\\ &\quad - J(\phi + i\tau + 2i\lambda) - J(2i\lambda + i\tau - \phi)\end{aligned} \tag{C47}$$

Using (C38), we see that this is the definition (33) of $Y(\phi)$.

To obtain (39c), we use the quasiperiodic condition (22) together with (C11a) to establish the formula

$$Q(\phi) = (-1)^{n+\nu} e^{-2n\tau - 3in\phi/2} F(\phi - 2i\tau)\, G(\phi - 4i\tau) \tag{C48}$$

We use this formula for $Q(\phi + 2i\lambda)$ in (29), and (C11b) for $Q(\phi - 2i\lambda)$. Using (C36) and (C40) as before, we obtain (39c) with

$$\begin{aligned} Z(\phi) = 1 + 2\pi[&J(\phi + 2i\lambda - i\tau) + J(2i\lambda - i\tau - \phi) \\ &- J(\phi + 3i\tau - 2i\lambda) - J(3i\tau - 2i\lambda - \phi)] \end{aligned} \tag{C49}$$

Using (C38) and (30b), this is the definition (34).

This completes the derivation of Eqs. (39a)–(39b) for $r(\phi)$. In Section 5 of the text we show that these results are consistent with the assumptions

$$\begin{aligned} &r(\phi) \to 0 \quad \text{as} \quad N \to \infty \quad \text{when} \quad 0 < \text{Im}(\phi) < \min(2\lambda, \tau) \\ &r(\phi) \to \infty \quad \text{as} \quad N \to \infty \quad \text{when} \quad -\min(2\lambda, \tau) < \text{Im}(\phi) < 0 \end{aligned} \tag{C50}$$

This checks Assumption B and the assumptions we used to obtain (C39) and (C41).

To obtain an equation for $T(\phi)$, we multiply (C18) by (C28) and find, using (C21),

$$\begin{aligned} T(\phi) = (-1)^\nu \xi^N &A(\phi - i\lambda + 2i\tau)\, A(2i\tau - i\lambda - \phi)\, F(\phi + 2i\lambda)\, G(\phi - 2i\lambda) \\ &\times P_1(\phi)\, P_4(\phi)/[F(\phi + 2i\tau)\, G(\phi - 2i\tau)] \end{aligned} \tag{C51}$$

First consider the leading behavior when $N \to \infty$. To do this, we note from (C2), (C24), and (C50) that $P_1(\phi)$, $P_4(\phi) \to 1$ as $N \to \infty$, provided $|\text{Im}(\phi)| < \min(2\lambda, \tau)$. Using also (C39), (C41), and (C13), we see from (C51) that

$$T(\phi) \sim (-1)^\nu t^N(\phi) \tag{C52}$$

provided

$$|\text{Im}(\phi)| < \min(2\lambda, \tau) \tag{C53}$$

The function $t(\phi)$ is independent of N and is given by

$$t(\phi) = \xi t_+(\phi)\, t_+(-\phi) \tag{C54}$$

where, if

$$h_+(\phi) = \prod_{m=0}^{\infty} (1 - q^m e^{i\phi}) \tag{C55}$$

then

$$t_+(\phi) = \prod_{m=0}^{\infty} \frac{h_+[\phi + 2i\tau + (4m - 1)\, i\lambda]\, h_+[\phi + (4m + 3)\, i\lambda]}{h_+[\phi + 2i\tau + (4m + 1)\, i\lambda]\, h_+[\phi + (4m + 5)\, i\lambda]} \tag{C56}$$

Note that $t(\phi)$ is analytic and nonzero in the domain (C53), which is in agreement with Assumptions B and C.

Using the relation between c and ξ given in Eq. (D36) of Ref. 1, we can express $c^{-1}t(\phi)$ as a product of terms of the form $1 - \alpha$, where $|\alpha| < 1$. Taking logarithms, Taylor-expanding each $\ln(1 - \alpha)$ term, and interchanging summations, we find that $t(\phi)$ is given by (30a).

Now take N to be finite and substitute the full expressions (C2), (C24), (C36), and (C40) for P_1, P_4, F, and G into (C51). Choosing $R_1 = R_2 = -R_3 = -R_4 = R$ and using (C46), we obtain

$$\ln[(-1)^\nu T(\phi)/t^N(\phi)] = \int_C \{\ln[1 + r(\phi')]\}\{W(\phi - \phi') - W(\phi + \phi')\}\, d\phi' \tag{C57}$$

where C is the line segment $(iR - \pi, iR + \pi)$ and

$$\begin{aligned} W(\phi) &= J(\phi + 2i\lambda) - J(2i\lambda - \phi) + J(2i\tau - \phi) - J(\phi + 2i\tau) - U(\phi) \\ &= (4\pi)^{-1}[\mathrm{Dn}(\phi + i\lambda) - 1] \end{aligned} \tag{C58}$$

using (C38), (C4), and (30b).

In this case we do lose some generality by choosing $R_1 = R_2$ and $R_3 = R_4$, since (C57) is only valid if

$$|\operatorname{Im}(\phi)| < \min(R, 2\lambda - R) \tag{C59}$$

Thus the maximum domain of validity is obtained by choosing $R = \lambda$. This is sufficient for considering the P.R., where $\phi = i\alpha$ and $|\alpha| < \lambda$ [Eqs. (16) and (17)]. Further, it turns out to be a convenient choice mathematically, since when $0 < \lambda < 2\tau/3$ to leading order for N large, $r(\phi) \sim p(\phi)$ on C and $p(\phi)$ has a saddle point at $\phi = i\lambda + \pi$ that dominates the r.h.s. of (C57).

Choosing therefore $R = \lambda$ and using (C58), together with the relations

$$\mathrm{Dn}(\phi) = \mathrm{Dn}(-\phi) = -\mathrm{Dn}(\phi + 2i\lambda) \tag{C60}$$

Eq. (C57) becomes the result (37) quoted in the text. It is apparent from (37) that $T(\phi)$ is an even function.

APPENDIX D. σ FOR $\lambda > 2\tau/3$

As we remark in Section 5, the derivation of the interfacial tension σ given in that section breaks down when $2\tau/3 < \lambda < \tau$.

To overcome this, let $r_0(\phi)$ and $r_1(\phi)$ be the functions $r(\phi)$ evaluated for $\nu = 0$ and 1, respectively. Then from (37)

$$\ln(-\Lambda_0/\Lambda_1) = (2\pi)^{-1} \int_{-\pi}^{\pi} B(i\lambda + u)[\mathrm{Dn}(u - i\alpha) + \mathrm{Dn}(u + i\alpha)]\, du \tag{D1}$$

where we have replaced ϕ by $i\alpha$, ϕ' by $i\lambda + u$

$$B(\phi) = \tfrac{1}{2} \ln\{[1 + r_0(\phi)]/[1 + r_1(\phi)]\} \tag{D2}$$

Thus we need to calculate the leading nonzero contribution to $B(\phi)$ when N is large and $\mathrm{Im}(\phi) = \lambda$. We remark that one way to tackle this problem that is almost certainly doomed to failure is to attempt to obtain explicit expansions of $r_0(\phi)$ and $r_1(\phi)$ as sums of terms that decrease exponentially with N. It appears that as $\lambda \to \tau$, a larger and larger number of dominant terms in the two expansions are the same, making it necessary to go to a large number of terms to find $B(\phi)$.

Instead we work with two functions $L(\phi)$ and $M(\phi)$ defined by

$$L(\phi) = \tfrac{1}{2}[r_0(\phi) + r_1(\phi)], \qquad M(\phi) = \tfrac{1}{2}[r_0(\phi) - r_1(\phi)] \tag{D3}$$

We consider only $\lambda > 2\tau/3$ and define two domains D_1 and D_2 by

$$D_1: \quad 0 < \mathrm{Im}(\phi) < 2(\tau - \lambda) \tag{D4a}$$

$$D_2: \quad 2(\tau - \lambda) < \mathrm{Im}(\phi) < \lambda \tag{D4b}$$

Then from (41c), when $\phi \in D_2$,

$$L(\phi) \sim p(\phi)\, p(\phi - 2i\tau) \tag{D5}$$

for N large, and $M(\phi)$ must be exponentially smaller than $L(\phi)$. Thus

$$1 \gg |L(\phi)| \gg |M(\phi)| \tag{D6}$$

In both D_1 and D_2 we know that $r_0(\phi)$ and $r_1(\phi)$ are exponentially small compared with unity. Thus

$$|1 + L(\phi)| \gg |M(\phi)| \tag{D7}$$

and from (D2) and (D3) we see that

$$B(\phi) \sim M(\phi) \tag{D8}$$

We therefore need to calculate $M(\phi)$ for $\mathrm{Im}(\phi) = \lambda$, or more generally for $\phi \in D_2$. When $\phi \in D_1$ we see from (41a) and (D3) that

$$M(\phi) \sim (-1)^n p(\phi) \tag{D9}$$

We start by assuming this result applies also in D_2: Less strongly, we assume that if M_b is the least upper bound of $|M(i\lambda + u)|$ for u real, then

$$\lim_{N\to\infty} [M_b/k_2{}^n] \tag{D10}$$

exists. We then calculate $M(\phi)$ and verify this assumption.

We use (39c) to obtain a recursion relation for $M(\phi)$ when N is large. To do this, we choose a ϕ in D_2, then choose an R so that the restrictions governing (39c) are satisfied, then move the contour C to the line segment $(i\lambda - \pi, i\lambda + \pi)$. The contour C is thereby forced to cross two poles in the complex ϕ' plane of the function

$$f(\phi') = Z(\phi - \phi' - i\tau) - Z(\phi + \phi' - i\tau) \tag{D11}$$

the poles occurring when

$$\phi - \phi' - i\tau = i(\tau - 2\lambda), \qquad \phi - \phi' - i\tau = -i\tau \tag{D12}$$

Both have residue $-i$.

Taking account of the contributions of these poles, Eq. (39c) becomes

$$\ln\{r(\phi)/[p(\phi)p(\phi - 2i\tau)]\} = \ln[1 + r(\phi - 2i\tau + 2i\lambda)] + \ln[1 + r(\phi)] + (2\pi)^{-1} \int_C \{\ln[1 + r(\phi')]\} f(\phi')\, d\phi' \tag{D13}$$

provided $\phi \in D_2$, where C is now the line segment $(i\lambda - \pi, i\lambda + \pi)$.

This equation holds for both $\nu = 0$ and $\nu = 1$. Taking the difference of the equations for the two cases, using (D3), (D6), and (D7), and retaining only dominant terms, we find that

$$M(\phi)/L(\phi) = M(\phi - 2i\tau + 2i\lambda) + (2\pi)^{-1} \int_C M(\phi') f(\phi')\, d\phi' \tag{D14}$$

for $\phi \in D_2$. We have neglected a term $M(\phi)$ on the r.h.s., since it must be negligible compared with the l.h.s.

The function $f(\phi')$ is continuous on C and independent of N. Thus from (D10) the integral in (D14) is of order $k_2{}^n$.

The function $L(\phi)$ in (D14) is known, being given by (D5). Using the periodicity property $p(\phi)\, p(\phi + 2i\lambda) = 1$, (D5) can also be written

$$L(\phi) = p(\phi)/p(\phi - 2i\tau + 2i\lambda) \tag{D15}$$

Also, when $2(\tau - \lambda) < \text{Im}(\phi) < 4(\tau - \lambda)$, the function $M(\phi - 2i\tau + 2i\lambda)$ in (D14) is given by (D9). Thus in this case solving (D14) for $M(\phi)$ gives

$$M(\phi) \sim (-1)^n p(\phi) + L(\phi)\, \mathcal{O}(k_2{}^n) \tag{D16}$$

which is virtually the same form as (D9). If necessary, we can now use this result to solve (D14) for $M(\phi)$ when $4(\tau - \lambda) < \text{Im}(\phi) < 6(\tau - \lambda)$, and so on. In all cases we find $M(\phi)$ is given by (D16) in the domain D_2.

Further, if we allow ϕ to cross the $\text{Im}(\phi) = \lambda$ boundary of D_2, the only effect is to replace $\ln[1 + r(\phi)]$ in (D13) by $-\ln[1 + r(2i\lambda - \phi)]$. In either case these terms are negligible and do not affect (D14). Thus (D16) is valid also for $\text{Im}(\phi) = \lambda$.

The point we wish to make is that the second term on the r.h.s. of (D16) is in general exponentially small compared with the first. Hence we can use the form (D9) of $M(\phi)$ in (D8) and (D1), giving the same equations (49) and (53) for the interfacial tension as apply when $0 < \lambda < 2\tau/3$.

More precisely, from (50), $p(\phi)$ has a saddle point at $\phi = i\lambda + \pi$, its absolute value there being $k_2{}^n$. Since $|L(\phi)| \ll 1$ for $\text{Im}(\phi) = \lambda$, the second term on the r.h.s. of (D16) is exponentially small compared with this saddle-point value, and to first order can be neglected in the integrands of (D1) and (D14).

This verifies our original assumption (D10) and our above remarks on the interfacial tension.

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